Counting Methods

Counting principle, permutations, combinations, probabilities

Part 1: The Fundamental Counting Principle

The Fundamental Counting Principle is the idea that if we have \( a \) ways of doing something and \( b \) ways of doing another thing, then there are \( a \cdot b \) ways of performing both actions.

Ex: When you decide to order pizza, you must first choose the type of crust: thin or deep dish (2 choices). Next, you choose the topping: cheese, pepperoni, or sausage (3 choices).

Using the rule of product, you know that there are \( 2 \times 3 = 6 \) possible combinations of ordering a pizza.

The tree diagram may be useful in working with the counting principle.

Ex: If a coin is tossed and the number cube is rolled simultaneously then the tree diagram can be the following:

We see that the total number of possibilities is \( 2 \times 6 = 12 \).

The number of ways in which a series of successive things can occur is found by multiplying the number of ways in which each thing can occur.

Ex: The number of possible outfits from 2 pairs of jeans, 3 T-shirts and 2 pairs of sneakers are: \( 2 \cdot 3 \cdot 2 = 12 \)
Ex: Next semester, you are planning to take three courses – math, English and humanities. There are 10 sections of math, 8 of English, and 5 of humanities that you find suitable. Assuming no scheduling conflicts, how many different three-course schedules are possible?

There are \(10 \cdot 8 \cdot 5 = 400\) different three-course schedules.

Ex: You are taking a multiple-choice test that has eight questions. Each of the questions has four answer choices, with one correct answer per question. If you select one of these four choices for each question and leave nothing blank, in how many ways can you answer the questions?

The number of different ways you can answer the questions is:

\[
4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^8 = 65,536.
\]

Ex: Telephone numbers in the United States begin with three-digit area codes followed by seven-digit local telephone numbers. Area codes and local telephone numbers cannot begin with 0 or 1. How many different telephone numbers are possible?

This situation involves making choices with ten groups of items. Here are the choices for each of the ten groups of items:

<table>
<thead>
<tr>
<th>Area Code</th>
<th>Local Telephone Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>10 10 10</td>
</tr>
<tr>
<td>8</td>
<td>10 10 10 10 10 10 10</td>
</tr>
</tbody>
</table>

The total number of different telephone numbers is:

\[
8 \cdot 10 \cdot 10 \cdot 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,400,000,000
\]

Part 2: Permutations

Permutation is an ordered arrangement of items that occurs when:
– No item is used more than once.
– The order of arrangement makes a difference.

Ex: There are six permutations of the set \{1,2,3\}, namely (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1).

If \( n \) is a positive integer, the notation \( n! \) (read “\( n \) factorial”) is the product of all positive integers from \( n \) down through 1.

\[
n! = n(n-1)(n-2)\cdots(3)(2)(1)
\]

\( 0! \) (zero factorial), by definition, is 1:

\[
0! = 1
\]

Ex: You need to arrange eight of your favorite books along a small shelf. How many different ways can you arrange the books, assuming that the order of the books makes a difference to you?

You can choose any one of the eight books for the first position on the shelf. This leaves seven choices for the second position. After the first two positions are filled, there are six books to choose and so on.

\[
8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40,320.
\]

If \( n \) is a positive integer, then \( (n + 1)! = n! \cdot (n + 1) \).

Ex:

a. \[
\frac{25!}{24!} = \frac{24! \cdot 25}{24!} = 25.
\]

b. \[
\frac{10!}{7!} = \frac{7! \cdot 8 \cdot 9 \cdot 10}{7!} = 8 \cdot 9 \cdot 10 = 720.
\]

If we have a set of \( n \) objects and we want to select \( r \) of them and arrange them in order, the total number of possibilities is
\[ nP_r = \frac{n!}{(n-r)!} \]

and is called the number of possible **permutations of \( r \) items are taken from \( n \) items**.

Ex: From a class of 20 students we need to select 3 for a committee, one to be president, another one to be vice-president and the third one to be secretary.

In this case \( n = 20 \), and \( r = 3 \). The order is important, so we apply the formula

\[ _{20}P_3 = \frac{20!}{(20-3)!} = \frac{17! \cdot 18 \cdot 19 \cdot 20}{17!} = 18 \cdot 19 \cdot 20 = 6840. \]

**Permutations of duplicate items**: the number of permutations of \( n \) items, where \( n_1 \) items are identical, \( n_2 \) items are identical, \( n_3 \) items are identical, and so on, is given by:

\[ \frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot ...} \]

Ex: In how many distinct ways can the letters of the word MISSISSIPPI be arranged?

The word contains 11 letters (\( n = 11 \)) where four Is are identical (\( n_1 = 4 \)), four Ss are identical (\( n_2 = 4 \)) and 2 Ps are identical (\( n_3 = 2 \)). The number of distinct permutations is:

\[ \frac{11!}{4! \cdot 4! \cdot 2!} = \frac{4! \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}{4! \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 2} = 34,650. \]

**Part 3: Combinations**

A **combination** is a way of selecting several things out of a larger group, where (unlike permutations) order does not matter.

A combination of items occurs when
• The items are selected from the same group.
• No item is used more than once.
• The order of items makes no difference.

Remarks:  
**Permutation** problems involve situations in which **order matters.**

**Combination** problems involve situations in which the **order** of items **makes no difference.**

The number of possible combinations if \( r \) items are taken from \( n \) items is:

\[
\binom{n}{r} = \frac{n!}{r! \cdot (n - r)!}.
\]

Remark: \( \binom{n}{r} < n^r \), for any \( r > 0 \).

Ex: From a class of 20 students we need to select 3 for a committee.

In this case \( n = 20 \), and \( r = 3 \). The order is not important, so we apply the formula

\[
\binom{20}{3} = \frac{20!}{3! \cdot (20 - 3)!} = \frac{17! \cdot 18 \cdot 19 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 17!} = 1,140.
\]

**Part 4: Basics of Probability**

*Probability* is used to describe an attitude of mind towards some proposition of whose truth we are not certain.

The basic concepts related to probability are:

• **Experiment** is any occurrence for which the outcome is uncertain.
• **Sample space** is the set of all possible outcomes of an experiment, denoted by \( S \).
• **Event**, denoted by \( E \) is any subset of a sample space.
• **Sum** of the theoretical probabilities of all possible outcomes is 1.
If an event $E$ has $n(E)$ equally likely outcomes and its sample space $S$ has $n(S)$ equally-likely outcomes, the theoretical probability of event $E$, denoted by $P(E)$, is:

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)}.$$ 

The value of the probability can be expressed as a ratio, as a decimal number or as percent. In a number format, always $0 \leq P(E) \leq 1$ and in percent format always $0\% \leq P(E) \leq 100\%$.

Remarks: Probabilities are assigned values from 0 to 1.

The closer the probability of a given event is to 1, the more likely it is that the event will occur.

The closer the probability of a given event is to 0, the less likely that the event will occur.

Ex: What is the probability to find a family with 2 boys among the all families having three children?

If we denote $B =$ boy and $G =$ girls, then the sample space for this experiment $S$ will be $S = \{\text{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG}\}$, where BGB means that the first child is boy, the second is girl and the third is boy. It follows that $n(S) = 8$ (the total number of outcomes).

The event $E$ is to find a family with two boys, so that $E = \{\text{BBG, BGB, GBB}\}$ and $n(E) = 3$.

Based on the probability formula, we can conclude that the chance to find/select families with exactly two boys among the set of all families with three children is:

$$P(E) = \frac{3}{8} = 0.375 = 37.5\%.$$ 

Part 5: Probability with the Fundamental Counting Principle, Permutations, and Combinations
Ex: A lottery game, LOTTOPLUS, is set up so that each player chooses seven different numbers from 1 to 58. With one LOTTOPLUS ticket, what is the probability of winning the prize?

Because the order of the seven numbers does not matter, this situation involves combinations:

\[ P(\text{Winning}) = \frac{\text{number of ways of winning}}{\text{total number of possible combinations}}. \]

We have that the number of ways of winning is 1 (one ticket) and the total number of possible combinations is given by:

\[ 58C_7 = \frac{58!}{7!(58-7)!} = \frac{51! \cdot 52 \cdot 53 \cdot 54 \cdot 55 \cdot 56 \cdot 57 \cdot 58}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 51!} = 300,674,088. \]

The probability to win the price is:

\[ P(\text{Winning}) = \frac{1}{300,674,088} = 0.00000003326 = 0.0000003326\%. \]

Ex: Two cards are drawn at random from a standard deck of 52 cards, without replacement. What is the probability that both cards drawn are queens?

\[ P(E) = \frac{\text{the way to draw 2 cards out of possible 4 queens}}{\text{the way to draw 2 cards from a deck of 52}} = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{12}{2652} = \frac{1}{221} = 0.0045 = 0.45\%. \]